THE STABLE FORM OF EQUILIBRIUM OF A PEBBLE BEACH SUBJECTED TO SURF

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When subjected to wave action, a beach composed of pebble or sand assumes the form of a cylindrical surface which remains fixed for constant wave forces. If the wave force changes, this surface will moove.

The present paper gives an approximate quantitative analysis of this phenomenon and compares the results with observational data. The consistancy of position of the particles of the beach in a stable state is, apparently, guaranteed by the condition of equilibrium of the gravity force mg, the hydrodynamic resistance P_h, and the force of interaction with the adjacent particles P_f (Fig.1), i.e.

$$P_h - P_f + mg \sin \alpha = 0 \tag{1}$$

It can be considered that

$$P_f = mgf \cos \alpha, \qquad P_h = {}^1/_2\, \rho v_{\max}{}^2 S C_x$$

where f is the coefficient of friction, $v_{\max}$ is the maximum velocity of the water flowing around the particle in the backrush phase.

The equation of the curve $y = y(x)$ which describes the stable form of the beach is unknown; therefore, α is an unknown quantity. The magnitude of $v_{\max}$ is also unknown. To determine $v_{\max}$ it is necessary to solve the hydrodynamic problem of the motion of the sheet of water of the surf on the sloping beach. Inasmuch as this sheet of water usually proves to be a thin one, its motion may be described quite simply by approximate hydrodynamic relations. If we introduce the coordinate s along the curve $y = y(x)$ and n along the normal (Fig.1), the approximate equations will have the form

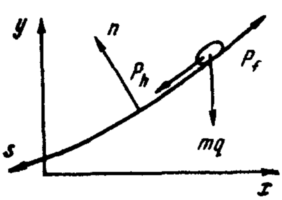

Fig. 1

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial s} = -\frac{g}{2n}\frac{\partial}{\partial s}(n^2 \cos \alpha) + g \sin \alpha - F_f$$

$$\frac{\partial n}{\partial t} + \frac{\partial (vn)}{\partial s} = 0 \tag{2}$$

where v is the velocity along s, n is the thickness of the sheet, and F_f is the resisting force per unit mass of water due to the rough bottom. The width of an element of fluid and its length along s are denoted by b and ds, respectively. Then assuming that the pebble particles are oblate spheroids with major axis a and minor axis c, we may conclude that

$\sim bds/a^2$ particles of pebble will be located in the area bds occupied by an elementary volume of fluid. Therefore, the force exerted by the area bds of the bottom on the volume of fluid of mass $dm = \rho n b ds$ will be equal to $\rho v^2 S C_x b ds / 2a^2$, and the force acting per unit mass will be

$$F_f = \frac{v^2 S C_x}{2a^2 n} \tag{3}$$

where S is that area of the piece of pebble which takes part in forming the hydrodynamic resistance, and C_x is the corrficient of this resistance.

To estimate the various terms of Equations (2) and to simplify the latter, we introduce the dimensionless variables V, σ, h, τ in accordance with Formulas

$$v = \sqrt{gl}\,V, \quad s = l\sigma, \quad n = \varepsilon l h, \quad t = \sqrt{l/g}\,\tau \tag{4}$$

where l is a characteristic length along s and $\varepsilon \ll 1$. Then from obvious physical considerations it follows that $V \sim \sigma \sim h \sim \tau \sim 1$. The Equations (2) transform into

$$\frac{\partial V}{\partial \tau} + V\frac{\partial V}{\partial \sigma} = -\frac{\varepsilon}{2h}\frac{\partial (h^2 \cos\alpha)}{\partial \sigma} + \sin\alpha - \frac{C_x \pi c \zeta}{\varepsilon 8 a h} V^2, \quad \frac{\partial h}{\partial \tau} + \frac{\partial (Vh)}{\partial \sigma} = 0 \tag{5}$$

where the Expression $\frac{1}{4}\pi a c \zeta$ is substituted for the area S $(\zeta < 1)$. The coefficient ζ takes account of the fact that a pebble particle does not project completely above the adjacent ones and the effective midsection area in the formula for hydrodynamic resistance should be taken smaller than the actual midsection area.

It is apparent from the Equations (5) that the first term on the right-hand side of the first equation can be neglected and that the last term is of the order of $0.005/\varepsilon$ for the natural estimates $C_x \sim 0.1$, $\zeta c/a \sim 0.1$ (flattened pebble). That is, the conditions of applicability of these relations is $\varepsilon \gtrsim 0.005$. We then have the simplified equations

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial s} = g\sin\alpha - \frac{C_x \pi \zeta c}{8an} v^2, \qquad \frac{\partial n}{\partial t} + \frac{\partial (vn)}{\partial s} = 0 \tag{6}$$

Transforming to the Lagrangian coordinates s_0, t $(s = s(s_0, t),\ t = t)$, we obtain

$$\frac{\partial s}{\partial t} = v, \quad \frac{\partial v}{\partial t} = g\sin\alpha - \frac{k}{n}v^2, \quad n\frac{\partial s}{\partial s_0} = f(s_0), \quad k = \frac{\pi C_x \zeta c}{8a} \tag{7}$$

where $f(s_0)$ is an arbitrary function which is determined from the inital data.

Equations (7) must be supplemented by the initial conditions

$$v(s_0, 0) = v_0(s_0), \quad n(s_0, 0) = n_0(s_0), \quad s(s_0, 0) = s_0$$

Observation shows that at the instant of the beginning of the backrush of the wave, the velocities of all particles of the water go to zero almost simultaneously, i.e. the condition $v_0(s_0) = 0$ can be assumed. The solution of the problem which has arisen is complicated, for the addition to its non-linearity, $\alpha(s_0)$ is unknown and must be determined from Equation (1) which contains, in turn, the unknown v_{max} found from the solution of the hydrodynamic problem.

However, for our purposes, there is no necessity to solve the entire hydrodynamic problem; it is sufficient to determine v_{max} for every point of the curve over which the sheet of water flows. This last problem can be solved approximately in the following manner.

In Equations (1) and (7) let us transform to the dimensionless variables

$$v = \sqrt{ga}\,V, \quad t = \sqrt{a/g}\,\tau, \quad s = a\sigma, \quad s_0 = a\sigma_0, \quad n = a\nu \tag{8}$$

and take account of Formula

$$m = {}^1/_6\,\pi a^2 c\,(\rho_1 - \rho) \tag{9}$$

where ρ_1 is the density of the material of the pebble. We shall then have

$$\frac{\partial\sigma}{\partial\tau} = \nu, \qquad \frac{\partial V}{\partial\tau} = \sin\alpha - \frac{p}{\nu}V^2 \tag{10}$$

$$\sin\alpha - f\cos\alpha + qV_{\max}^2 = 0 \tag{11}$$

$$p = \frac{\pi\zeta C_x c}{8a}, \qquad q = \frac{3\zeta C_x \rho}{4(\rho_1 - \rho)} \tag{12}$$

If in Equations (10) we transform from the variables σ_0, τ to σ_0, σ, then the equation which is obtained for V^2 (a linear ordinary differential equation) can be formally integrated, taking account of the initial condition $V_0 = 0$

$$V^2 = 2\int_{\sigma_0}^{\sigma} \sin\alpha(\xi)\exp\left[-\int_{\xi}^{\sigma}\frac{2p\,d\lambda}{\nu(\sigma_0, \lambda)}\right]d\xi \tag{13}$$

This integral is formal since $\nu(\sigma_0, \lambda)$ is unknown. However, Equation (13) permits us to make estimates necessary for the determination of $V_{\max}$. We differentiate (13) with respect to σ_0

$$\frac{\partial V^2}{\partial\sigma_0} = -2\sin\alpha(\sigma_0) + 2\int_{\sigma_0}^{\sigma}\sin\alpha(\xi)\exp\left[-\int_{\xi}^{\sigma}\frac{2p\,d\lambda}{\nu(\sigma_0, \lambda)}\right]\left[\int_{\xi}^{\sigma}\frac{2p}{\nu^2(\sigma_0\lambda)}\frac{\partial\nu(\sigma_0\lambda)}{\partial\sigma_0}d\lambda\right]d\xi \tag{14}$$

From (14) we have $\partial V^2/\partial\sigma_0 < 0$ for $\sigma \sim \sigma_0$. As $\tau \to \infty$, i.e. as $\sigma \to \infty$, we have from (10), (11) and (13)

$$V \to V_\infty = \text{const}, \quad \nu \to \nu_\infty = \text{const}, \quad \alpha \to \alpha_\infty = \text{const}, \quad \nu_\infty \sin\alpha_\infty - pV_\infty^2 = 0$$

i.e. ν eventually ceases to depend on σ_0 and the second term on the right-hand side of (14) goes to zero. This allows us to conclude with a great degree of certainty that the inequality $\partial V^2/\partial\sigma_0 > 0$ never occurs, that is, that for each value of σ the maximum velocity is attained at the fluid particle $\sigma_0 = 0$.

Thus, for every point on the curve we have $V_{\max} = V|_{\sigma_0}$ Therefore, the problem finally reduces to the solution of the following system of equations and initial conditions (the subscript max on V is omitted)

$$\frac{d\sigma}{d\tau} = V, \quad \frac{dV}{d\tau} = \sin\alpha(\sigma) - \mu V^2, \quad \sin\alpha - f\cos\alpha + qV^2 = 0, \quad \sigma(0) = 0, \quad V(0) = 0 \tag{15}$$

We shall assume that the quantity $\mu = p/\nu$ in these equations is constant, for ν varies only slightly and some mean value ν_* can be used. This mean value can be detrmined, for example, experimentally from the relation

$$\left(\frac{q}{\mu} + 1\right)\sin\alpha_\infty = f\cos\alpha_\infty \tag{16}$$

which is obtained from (15) for $\tau \to \infty$, $dV/d\tau \to 0$. If the quantity α_∞ is known from experiment, then ν_* and the ratio q/μ are obtained in accordance with (16)

$$\frac{q}{\mu} = \frac{f}{\tan\alpha_\infty} - 1; \quad \nu_* = \frac{\pi c(\rho_1 - \rho)}{6a\rho}\left(\frac{f}{\tan\alpha_\infty} - 1\right) \tag{17}$$

Eliminating the quantity V from (15), we obtain an equation for α alone

$$\frac{d\alpha}{d\tau} = -\frac{2\mu}{\sqrt{q}}\sqrt{f\cos\alpha - \sin\alpha}\,\frac{(1 + q/\mu)\sin\alpha - f\cos\alpha}{f\sin\alpha + \cos\alpha}, \qquad \alpha(0) \equiv \alpha_0 = \tan^{-1} f \tag{18}$$

We thus have an initial value problem for $\alpha(\tau)$, which is solvable by quadratures

$$\int_{\alpha_0}^{\alpha}\frac{(f\sin\alpha + \cos\alpha)\,d\alpha}{[(1 + q/\mu)\sin\alpha - f\cos\alpha]\sqrt{f\cos\alpha - \sin\alpha}} = -\frac{2\mu}{\sqrt{q}}\tau \tag{19}$$

Using the relations $d\sigma\cos\alpha = d\xi$ $(\xi = x/a)$, $d\sigma = V\,d\tau$ and Equations

(15), we find (20)

$$\frac{d\xi}{d\alpha} = -\frac{1}{2\mu}\frac{\cos\alpha\,(f\sin\alpha+\cos\alpha)}{(1+q/\mu)\sin\alpha - f\cos\alpha} \quad \text{or} \quad \xi = -\frac{1}{2\mu}\int_{\alpha_0}^{\alpha}\frac{\cos\alpha\,(f\sin\alpha+\cos\alpha)\,d\alpha}{(1+q/\mu)\sin\alpha - f\cos\alpha}$$

Furthermore, $d\eta = -\tan\alpha\, d\xi$, so that

$$\eta = \frac{1}{2\mu}\int_{\alpha_0}^{\alpha}\frac{\sin\alpha\,(f\sin\alpha+\cos\alpha)\,d\alpha}{(1+q/\mu)\sin\alpha - f\cos\alpha} \qquad (21)$$

We introduce the variables $2\mu\xi = X$, $2\mu\eta = Y$. Then

$$X = \int_{\alpha}^{\alpha_0}\frac{\cos\alpha\,(f\sin\alpha+\cos\alpha)\,d\alpha}{(1+q/\mu)\sin\alpha - f\cos\alpha} =$$

$$= \frac{\sin\alpha_\infty}{\sin\alpha_0}\left\{\cos\alpha_\infty - \cos(\alpha+\alpha_\infty-\alpha_0) + \cos\alpha_\infty\cos(\alpha_0-\alpha_\infty)\ln\frac{\tan[{}^1/_2(\alpha_0-\alpha_\infty)]}{\tan[{}^1/_2(\alpha-\alpha_\infty)]}\right\}$$

$$-Y = \int_{\alpha}^{\alpha_0}\frac{\sin\alpha\,(f\sin\alpha+\cos\alpha)\,d\alpha}{(1+q/\mu)\sin\alpha - f\cos\alpha} = \qquad (22)$$

$$= \frac{\sin\alpha_\infty}{\sin\alpha_0}\left\{\sin\alpha_\infty - \sin(\alpha+\alpha_\infty-\alpha_0) + \sin\alpha_\infty\cos(\alpha_0-\alpha_\infty)\ln\frac{\tan[{}^1/_2(\alpha_0-\alpha_\infty)]}{\tan[{}^1/_2(\alpha-\alpha_\infty)]}\right\}$$

Eliminating the parameter α , we obtain the equation of the curve $Y = -\Phi(X; f, f_\infty)$, which depends only on the two parameters f and $f_\infty = \tan\alpha_\infty$. In order to transform to the dimensional variables x and y , X and Y must be multiplied by the dimensional factor L

$$L = \frac{a}{2\mu} = \frac{4a^2 v_*}{\pi\zeta C_x c} = \frac{2(\rho_1-\rho)}{3\zeta C_x\rho}\left(\frac{f}{f_\infty}-1\right)a \qquad (23)$$

Substituting the values $\zeta \sim 0.3$, $C_x \sim 0.3$, $\rho_1/\rho \sim 3$, $f/f_\infty \sim 3$, we have $L \sim 30\,a$; if the dimension of the pebble $a \sim 3$ cm , then $L \sim 1$ m , which is in accord with the results of observation. If, however, we have a sand for which we may take $C_x \sim 0.6$, $\zeta \sim 0.3$, $\rho_1/\rho \sim 3$, $f/f_\infty \sim 7$, $a \sim 0.1$ cm, then we have $L \sim 4.5$ cm . Thus, for a sand beach the curve $y = y(x)$ is practically a straight line.

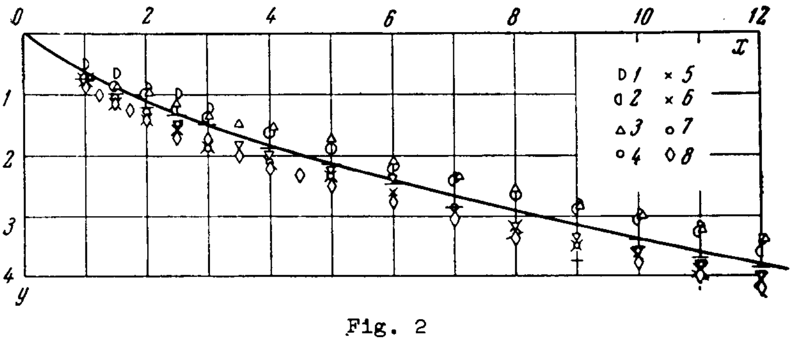

Fig. 2

In the summer of 1962, the author carried out some measurements of the form of the curve $y = y(x)$ on a pebble beach at the settlement of Rybach'ye (Crimea). The measurements were made at various surf intensity in calm and stormy weather, using the simplest apparatus. The results are shown in Fig.2. The symbols in Fig.2 corresond to the following conditions under which measurenents were conducted: triangles with vertex down, rhombuses, circles, horizontal lines, and lines inclines to the left or right are measurements in calm weather for light surf; triangles with vertex up and the two forms of semicircles are measurements for a very stormy sea, which shifted the boundary of the zone washed by the waves far (by several meters)

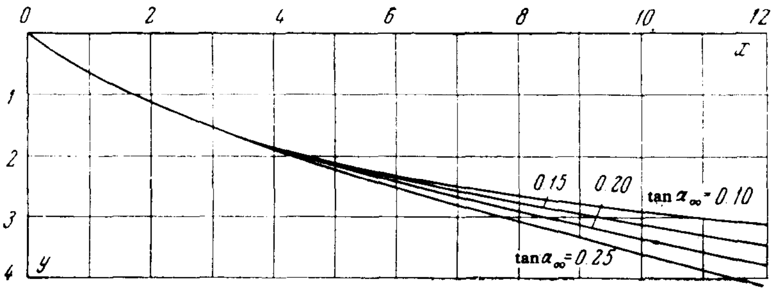

Fig. 3

onto the dry land. The curve of Equation (22) is shown on the same figure for the case $f = \tan\alpha_0 = 0.75$, $f_\infty = f/(1+q/\mu) = 0.20$, $\zeta = 0.30$, $C_x = 0.30$, $\rho_1/\rho = 3$, $a = 3$ cm. The unit of the coordinate axes is equal to 12 cm. Curves calculated from Equations (22) and (23) for the same conditions and the four values of f_∞ indicated are given in Fig.3.

To determine f properly, a cone of dry and wet pebble was poured on the beach and the angle of the vertex of the cone was measured. In addition, the angle of inclination of the curve $y = y(x)$ at the ridge ($x = 0$) was measured. The results of these measurements were close and gave $f = 0.75$. The choice of q/μ, i.e. of $f_\infty = \tan\alpha_\infty$ was made according to the experimental data for $y = y(x)$ for large x, where the curve $y = y(x)$ became a practically straight line (see Fig.2).

The data of Fig.2 indicate, first of all, that the curve $y = y(x)$ actually depends only slightly on the strength of the surf (as was mentioned above, the position of this curve, that is, the location of the point $x = y = 0$, depends greatly on the strength of the surf). Secondly, they show that the approximate theory which has been proposed describes the phenomenon satisfactorily.

A more careful examination of Fig.2 shows that the average experimental values of the ordinates of the curve $y = y(x)$ for stormy weather differ from the ordinates for a light surf by an approximately constant amount everywhere except for the first three values of the abscissa. Therefore, if the curve for heavy surf is shifted downward it will coincide with the curve for a light surf everywhere except at the points indicated. This deviation is explained as follows. While for a light surf the boundary between the washed and dry portions of the beach is outlined very clearly by a break in the cross-sectional profile, the boundary is more nearly washed away in a storm. This is related to the fact that the irregularity of a stormy surf is considerable and the edge of the sheet of water of the broken waves washes over the ridge of the wetted part of the beach (the point $x = y = 0$) and smoothes it out by carrying particles from the rigde to the dry area during the uprush of the wave, lowering the ordinates of the curve $y = y(x)$ in the vicinity of the point $x = 0$.

If the revision of the curve is made, the curve of Fig.2 computed by the proposed method turns out to be higher than the average of the experimental values. It is possible, however, to obtain coincidence of these curves by adjusting slightly the values of ζ and C_x assumed above, within the range of reasonable values for these quantities, i.e. by a suitable change in the scale factor L and the values of $\tan\alpha_\infty$.

We remark further that the equations which have been constructed are applicable for the description of only that portion of the beach which is periodically drained and then washed once more by the sheet of water from the waves. The profile of the portion of the beach which is always submerged is, as observation shows, steeper, and is inclined almost at the angle of repose. Thus the curve constructed above beginning at the place of impact of the breaking waves as they approach the beach gradually becomes steeper and steeper upon going out under the water and even at a depth of the order of 1 m slopes at an angle of repose.

The coordinates of the points of the curve $y = y(x)$ were measured with the aid of two graduated rods. The leveling of the rod Ox was carried out with the aid of an underwater swimming mask on the glass of which a small quantity of water was poured, making the mask into a level gage.

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Translated by A.R.R.

ON A STABILITY PROBLEM

(OB ODNOI ZADACHE USTOICHIVOSTI)

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In connection with the frequently passed, at the present time, scientific discussions on the subject of stability of elastic systems with follower forces we have programed and solved the following problem.

A thin elastic bar is executing a uniformly accelerated motion under the action of a follower force, applied at one of its ends.

The differential equation of the elastic line of a homogeneous bar will be

$$EI\frac{\partial^4 y}{\partial x^4} + \frac{\partial}{\partial x}\left[\frac{P}{l}(l-x)\frac{\partial y}{\partial x}\right] + \rho F\frac{\partial^2 y}{\partial t^2} = 0$$

Assuming $y = Xe^{i\Omega t}$ and passing to a nondimensional form we obtain

$$\eta^{IV} + \beta\,[(1-\zeta)\,\eta]' - \omega^2\eta = 0$$

Here

$$\beta = \frac{Pl^2}{EI}, \quad \omega^2 = \frac{\rho F l^4}{EI}\Omega^2, \quad \zeta = \frac{x}{l}, \quad \eta = \frac{X}{l}$$

The boundary conditions are

$$\eta'' = 0, \quad \eta''' = 0, \quad \text{for } \zeta = 0; \qquad \eta'' = 0, \quad \eta''' = 0 \quad \text{for } \zeta = 1$$

We seek a solution in the form of a series

$$\eta = A_0 + A_1\zeta + A_2\zeta^2 + A_3\zeta^3 + \ldots$$

According to the conditions at the ends

$$A_2 = A_3 = 0, \qquad \Sigma A_n n\,(n-1)\,(n-2) = 0, \qquad \Sigma A_n\, n\,(n-1) = 0 \tag{1}$$

we have for the determination of the terms of the series the recurrence formula